**Probability by Axioms and Rules**

**Axioms**

Let S be a sample space for a random experiment and A and B be events (sets of outcomes). The events A and B are **mutually exclusive** (ME) if A and B have no outcomes in common; i.e., A and B cannot occur simultaneously. The event **A or B** is the event consisting of all outcomes that are either in A or in B. The event **A and B** is the event that consists of all outcomes that are in both A and B. The event **notA** consists of all outcomes not in the event A.

1. 0 ≤ P(A) ≤ 1.
2. P(S) = 1.
3. (Sum Rule) If A and B are mutually exclusive then

P(A or B) = P(A) + P(B).

**Derived Rules**

1. (Complement Rule) P(not A) = 1 – P(A).
2. (General Sum Rule) P(A or B) = P(A) + P(B) – P(A and B).

**Example**: Carol has applied for admission to Harvard and to Princeton. The probability that Harvard accepts her is .3, the probability that Princeton accepts her is .4, and the probability that both accept her is .2. What is the probability that neither accept her?

6. (Equally Likely Rule) If the sample space consists of n equally likely outcomes,

then P(A) = .

**Examples**

1. Toss a fair die.

S =

P(die comes up prime) =

P(die comes up odd) =

1. Toss a fair coin 3 times.

S =

If X = number of heads, then the probability function for X is

x 0 1 2 3

p(x)

1. Toss a pair of dice.

S =

If X = dice sum, then the probability function for X is

x 2 3 4 5 6 7 8 9 10 11 12

p(x)

**Compare with the simulated result**

2 3 4 5 6 7 8 9 10 11 12

0.0298 0.0532 0.0776 0.1128 0.1347 0.1716 0.1435 0.1050 0.0837 0.0601 0.0280

P(X ≥ 10) =

P(X < 10) =

**Problem** I toss a fair coin twice. Assuming that at least one of the tosses produces a head, what is the probability that both tosses produced a head?

Let A = at least one head and B = both heads. The question asks for the conditional probability P(B|A) = probability of B given A.

**Conditional probability in general**

**7.** If P(A) ≠ 0, then

**Examples**

1. A pair of fair dice is tossed. Given that the sum of the dice is 7, what is the probability that one of the dice came up 3?
2. A group of 10 men and 15 women were polled as to whether they preferred eating an orange or eating an apple. The results of the poll are given below

|  |  |  |
| --- | --- | --- |
|  | **APPLE** | **ORANGE** |
| **MALE** | **4** | **6** |
| **FEMALE** | **10** | **5** |

M =male F = female A= apple O = orange

P(M) = P(F) = P(A) = P(O) =

**P(A | M) P(O | M)**

**P(A | F) P(F | A)**

8. (Product Rule) P(A and B) = P(A)P(B | A) (**Note**: This is true even if P(A) = 0.)

**Examples**

1. A jar contains 5 red and 4 green chips. Two chips are drawn without replacement . What is the probability that both are red? What is the probability that one is red and one is green?

**Using a tree diagram**

1. If the drawing in (a) is with replacement, what are the probabilities of the events in (a)?
2. Four cards are dealt from an ordinary deck of playing cards. What is the probability that they are all aces?
3. You have two jars of chips. Jar1 contains 4 red and 4 blue chips. Jar2 contains 5 red and 3 blue chips. You toss a fair coin. If the coin toss is heads, you draw a chip from Jar1. If it is tails, you draw from Jar2. What is the probability the you draw a red chip? What is the probability that if the drawn chip is red, it came from Jar1 (the coin was heads)?

9. (Law of Total Probability) P(A) = P(B)P(A | B) + P(not B)P(A | not B)

**Example**  A new test has been developed for a serious disease. Like all such tests it’s not perfect. It has a false positive rate of 2%; i.e., if a person doesn’t have the disease, there is a 2% chance that the test will be positive. It also has a false negative rate of 3%; i.e., if a person has the disease, there is a 3% chance that the test is negative. It is a relatively uncommon disease, affecting only 0.1% of the population. How good is the test? If you test positive, what’s the probability that you have the disease?

Basic Events: D = has the disease + = tests positive

Probability asked for: P(D | +)

Given information: P(D) = 0.001 P(+ | not D) = 0.02 P(- | D) = 0.03)

**Independence**

Two events A and B are independent if whether or not A occurs does not affect the probability that B occurs. Formally, A and B are **independent** if P(B) = P(B | A)..

**Suppose** a jar contains 3 red and 5 blue chips. Two chips are drawn from the jar. Let

A = first draw is red and B = second draw is red. If the drawing is done without replacement, A and B are not independent. The probability that B occurs depends on whether or not A occurred. If the drawing is done with replacement, A and B are independent.

**(10) Product Rule for Independent Events**

If A and B are independent, then P(A and B) = P(A)P(B).

**Example**  A fair die is tossed 3 times. What is the probability that a 1 occurs on each toss?

**Example**  A manufacturer claims that 99% of its product will still be functioning after 3 years. I buy 5 of the company’s product. What is the probability that all 5 of these items will still be functioning after 5 years?

**Exercises 4**

1. Suppose the disease in the example were not so rare. Suppose it were present in 1% of the population. If you tested positive, what would be the probability that you had the disease?
2. Suppose the disease in the example were common. Suppose it were present in 25% of the population. If you tested positive, what would be the probability that you had the disease?
3. Suppose a coin is biased and has a 25% chance of coming up heads. The coin is tossed twice.
4. What is the sample space?
5. Are the outcomes equally likely?
6. Use a tree diagram to compute the probabilities of the outcomes.
7. What is the probability of getting at least one head.
8. What is the probability of getting two heads given that at least one head occurs?
9. Quality control samples were taken for parts being produced by two assembly lines. A sample of 100 parts produced by assembly line 1 contained 2 defective parts. A sample of 200 parts produced by assembly line 2 contained 6 defective parts. The table below summarizes the results.

|  |  |  |
| --- | --- | --- |
|  | LINE 1 | LINE 2 |
| Defective | 2 | 6 |
| Non-defective | 98 | 194 |

1. Given that a part was found to be defective, what is the probability it came from Line 1?
2. Given that a part was found to be non-defective, what is the probability it came from Line 1?
3. A fair coin it tossed 5 times.
4. How many different outcomes are there?
5. Are these outcomes equally likely?
6. What is the probability that all the tosses produced heads?
7. What is the probability that at least one tall is produced?
8. What is the probability that exactly one head is produced?
9. Three cards are dealt from an ordinary deck of playing cards.
10. What is the probability that exactly one of the cards is a spade?
11. Let X = number of spades dealt. Find that probability function, p(x), for the random variable X.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **x** | **0** | **1** | **2** | **3** |
| **p(x)** |  |  |  |  |
|  |  |  |  |  |

1. A device is assembled from two primary parts. The first type of part is defective 3% of the time and the second type is defective 4% of the time. The device functions properly if and only if both of the parts are non-defective. What is the probability that a device made from these two parts will function properly? What are you assuming in making your calculation? Is your assumption reasonable?
2. A biased die is tossed two times. The probability for each face is given below.

Outcome 1 2 3 4 5 6

Probability .1 .3 .2 .2. .1. .1

1. What is the probability of getting two 1’s?
2. What is the probability of getting the same result on both tosses?